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Approximation Methods for Nonlinear m-Accretive Operator Equations*

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Suppose E is a real *reflexive* Banach space and $T: D(T) \subset E \to E$ is a *locally Lipschitzian m-accretive operator*, where the domain of T, $D(T)$, is a proper subset of E. For any $f \in E$, approximation methods are constructed which converge strongly to the solution of the equation $x + Tx = f$. Explicit error estimates are given and convergence is at least as fast a geometric progression.

INTRODUCTION

Let E be a real Banach space.

An operator T with domain $D(T)$ and range $R(T)$ in E is called *accretive* if the inequality

$$\|x - y\| \leq \|x - y + \lambda(Tx - Ty)\| \tag{1}$$

holds for all $x, y \in D(T)$ and for all $\lambda > 0$. If T is accretive and $(I + rT)(D(T)) = E$ for all $r > 0$ then T is called *m-accretive*, where I denotes the identity operator.

If T is m-accretive, then for any given $f \in E$ the equation

$$x + Tx = f \tag{2}$$

has a unique solution.

Recently, Chidume and the author [2] and Liang [4] studied methods for approximating the solution of Eq. (2) in *uniformly smooth Banach spaces*

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under the natural setting that the domain of T, $D(T)$, is a proper subset of E, T maps $D(T)$ into E, and is m-accretive. In this note we continue the study of methods of approximating the solution of Eq. (2) (when T is m-accretive) in Banach spaces much more general than those considered in Chidume and the author [2] and Liang [4] under the natural setting that the domain of T, $D(T)$, is a proper subset of E and T maps $D(T)$ into E. In particular, we extend Theorems 1 and 2 of Liang [4] from *real uniformly smooth Banach spaces* to the much more general *real reflexive Banach spaces*.

1. MAIN RESULTS

In the sequel L will denote the local Lipschitz constant of T.

THEOREM 1. *Supose E is a real reflexive Banach space, and $T : D(T) \subset E \to E$ is m-accretive and locally Lipschitzian. Suppose $D(T)$ is open and that $x^* \in D(T)$ is the unique solution of the equation $x + Tx = f$, $f \in E$. Suppose $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence satisfying the conditions*:

(i) $0 \leq \alpha_n \leq 1/2[L^2 + 2L + 2]$, $n \geq 0$

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then there exists a closed convex neighborhood B of x^* contained in $D(T)$ and for any given $x_0 \in B$, a sequence $\{x_n\}_{n=0}^{\infty}$ of elements of B such that on setting

$$p_n = (1 - \alpha_n)x_n + \alpha_n(f - Tx_n), \qquad n \geq 0$$

the sequence $\{p_n\}$ satisfies the condition

$$\|p_{n-1} - x_n\| = \inf\{\|p_{n-1} - x\| : x \in B\} \qquad \forall n \geq 1 \tag{3}$$

and converges strongly to x^*. Moreover, if $\alpha_n = 1/2[L^2 + 2L + 2]$ for all $n \geq 0$ then

$$\|p_n - x^*\| \leq \rho^n \|p_0 - x^*\|,$$

where $\rho = (1 - 1/4[L^2 + 2L + 2]) \in (0, 1)$.

Proof. Define $S : D(T) \to E$ by $Sx = f - Tx$. Observe that x^* is a fixed point of S and that S is locally Lipschitz with constant L. Furthermore $(-S)$ is accretive so that for all $r > 0$, and $x, y \in D(T)$

$$\|x - y\| \leq \|x - y - r(Sx - Sy)\|. \tag{4}$$

Without loss of generality we may assume $L \geq 1$. Let $B(y, r) = \{x \in E : \|x - y\| \leq r\}$. Then there exists $r_1 > 0$ such that $B(x^*, r_1) \subseteq D(T)$.

Since S is locally Lipschitzian, there exists $r_2 > 0$ such that S is Lipschitzian on $B(x^*, r_2)$. Let $r = \min\{r_1, r_2\}$. Then $B(x^*, r) \subseteq D(T)$ and S is Lipschitzian on $B(x^*, r)$. Let $B = B(x^*, r/2L)$. Given any $x_0 \in B$, then $\|Sx_0 - x^*\| \le r/2 < r$ so that $Sx_0 \in B(x^*, r)$. Hence $p_0 = (1 - \alpha_0)x_0 + \alpha_0 Sx_0 \in B(x^*, r)$. Since E is reflexive, there exists $x_1 \in B$ such that

$$\|p_0 - x_1\| = \inf\{\|p_0 - x\| : x \in B\}.$$

Thus

$$p_1 = (1 - \alpha_1)x_1 + \alpha_1 Sx_1 \in B(x^*, r).$$

By continuing this process we obtain $\{p_n\}$ in $B(x^*, r)$ and $\{x_n\}$ in B satisfying the conditions

$$\begin{aligned} p_n &= (1 - \alpha_n)x_n + \alpha_n Sx_n, \qquad n \ge 0 \\ \|p_{n-1} - x_n\| &= \inf\{\|p_{n-1} - x\| : x \in B\}, \qquad n \ge 1. \end{aligned} \tag{5}$$

Thus

$$\|x_n - x^*\| \le \|p_{n-1} - x^*\|, \qquad n \ge 1.$$

We now prove that $\lim p_n = x^*$. From (5) we obtain

$$\begin{aligned} x_n &= p_n + \alpha_n x_n - \alpha_n Sx_n \\ &= (1 + \alpha_n)p_n - \alpha_n Sp_n + \alpha_n^2(x_n - Sx_n) + \alpha_n(Sp_n - Sx_n). \end{aligned} \tag{6}$$

Observe that

$$x^* = (1 + \alpha_n)x^* - \alpha_n Sx^*. \tag{7}$$

Thus from (6) and (7) we obtain

$$\begin{aligned} \|x_n - x^*\| &= \|(1 + \alpha_n)(p_n - x^*) - \alpha_n(Sp_n - Sx^*) \\ &\quad + \alpha_n^2(x_n - Sx_n) + \alpha_n(Sp_n - Sx_n)\| \\ &\ge (1 + \alpha_n)\left\|p_n - x^* - \frac{\alpha_n}{1 + \alpha_n}(Sp_n - Sx^*)\right\| \\ &\quad - \alpha_n^2\|x_n - Sx_n\| - \alpha_n\|Sp_n - Sx_n\| \\ &\ge (1 + \alpha_n)\|p_n - x^*\| - \alpha_n^2\|x_n - Sx_n\| - \alpha_n\|Sp_n - Sx_n\| \\ &\qquad \text{(using (4))}. \end{aligned}$$

Hence

$$\begin{aligned}
\|P_n - x^*\| &\le \frac{1}{1+\alpha_n}\|x_n - x^*\| + \alpha_n^2\|x_n - Sx_n\| + \alpha_n\|Sp_n - Sx_n\| \\
&\le \left[1 - \alpha_n + \alpha_n^2\right]\|x_n - x^*\| \\
&\qquad + (1+L)\alpha_n^2\|x_n - x^*\| + L(1+L)\alpha_n^2\|x_n - x^*\| \\
&= \left[1 - \alpha_n + (L^2 + 2L + 2)\alpha_n^2\right]\|x_n - x^*\| \\
&\le \left[1 - \tfrac{1}{2}\alpha_n\right]\|p_{n-1} - x^*\| \\
&\le \exp\left(-\frac{1}{2}\sum_{j=0}^{n}\alpha_j\right)\|p_0 - x^*\| \to 0 \qquad \text{as } n \to \infty.
\end{aligned} \tag{8}$$

If we set $\alpha_n = 1/2[L^2 + 2L + 2]\ \forall n \ge 0$ in (8) we obtain

$$\|p_n - x^*\| \le \left[1 - \frac{1}{4(L^2 + 2L + 2)}\right]\|p_{n-1} - x^*\| \le \rho^n\|p_0 - x^*\|,$$

completing the proof of Theorem 1.

Remark 1. In proving Theorem 1 we first observed that the accretivity of T implies that $(-S)$ is accretive and hence inequality (4) holds. In order to apply inequality (4) we used (5) to express x_n in the form (6). Furthermore, we expressed x^* in the form (7). Using (6) and (7) and an application of inequality (4) and the Lipschitz condition of S we were able to prove Theorem 1 without the application of an inequality of Reich used in [4] and which hold in uniformly smooth Banach spaces. Moreover, we did not apply a lemma of Weng which was also used in [4].

Remark 2. Theorem 1 extends Theorems 1 and 2 of Liang [4] from real uniformly smooth Banach spaces to real reflexive Banach spaces. Moreover, our theorem provides additional information about the error estimates. Convergence is at least as fast as a geometric progression.

Remark 3. Using straightforward modifications of the proof of Theorem 1 one can easily prove the following theorem for a more general iteration scheme.

THEOREM 2. *Suppose $E, T, D(T), S$, and x^* are as in Theorem* 1. *Suppose $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real sequences satisfying the conditions*:

(i) $0 \le \alpha_n \le 1/2[L^2 + 2L + 2], \qquad n \ge 0$

(ii) $0 \le \beta_n \le 1/4[L^2(1+L)], \qquad n \ge 0$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty.$

Then there exists a closed convex neighborhood B of x^* contained in $D(T)$ and for any given $x_0 \in B$, a sequence $\{x_n\}_{n=0}^{\infty}$ of elements of B such that on setting

$$y_n = (1 - \beta_n)x_n + \beta_n S x_n, \qquad n \geq 0$$

$$p_n = (1 - \alpha_n)x_n + \alpha_n S y_n, \qquad n \geq 0$$

the sequence $\{p_n\}$ satisfies the condition

$$\|p_{n-1} - x_n\| = \inf\{\|p_{n-1} - x\| : x \in B\} \qquad \forall n \geq 1$$

and converges strongly to x^*. Moreover, if $\alpha_n = 1/2[L^2 + 2L + 2]\ \forall n \geq 0$, and $\beta_n = 1/4L^2(1 + L)\ \forall n \geq 0$ then

$$\|p_n - x^*\| \leq \rho^n \|p_0 - x^*\|,$$

where $\rho = (1 - 1/8[L^2 + 2L + 2]) \in (0, 1)$.

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REFERENCES

1. F. E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, *in* "Proc. Sympos. Pure Math.," Vol. 18, Part 2, Amer. Math. Soc., Providence, Rhode Island, 1976.
2. C. E. Chidume and M. O. Osilike, Approximation methods for nonlinear operator equations of the m-accretive type, *J. Math. Anal. Appl.* **189** (1995), 225–239.
3. T. Kato, Nonlinear semigroups and evolution equations, *J. Math. Soc. Japan* **19** (1967), 508–520.
4. Z. Liang, Iterative solution of nonlinear equations involving m-accretive operators in Banach spaces, *J. Math. Anal. Appl.* **188** (1994), 410–416.